Week 4 (disc 106)

Wednesday, February 10, 2021 10:49 AM

Math 128a – Week 4 Worksheet
GSI: Izak, (2/10/21)

2.3 Problems

Problem 1. *Come up with a function $f \in C^2[a,b]$ with $f(p) = 0$ for some $p \in [a,b]$ such that Newtons method fails to converge for any initial guess not equal to p.*

Problem 2. *Derive the error formula for Newton's method:*

$$|p - p_{n+1}| \leq \frac{M}{2|f'(p_n)|}|p - p_n|^2$$

2.4 Problems

Problem 3. *Generalize one of your homework problems. Construct a sequence p_n converging to p at order α with asymptotic error constant λ.*

2.5 Problems

Problem 4. *Steffensen's method is applied to a function $g(x)$ using $p_0^{(0)} = 1, p_2^{(0)} = 3$ to obtain $p_0^{(1)} = .75$. What is $p_1^{(0)}$?*

Problem 5. *Prove that if p_n converges linearly to p and $\lim_{n\to\infty} \frac{p_{n+1}-p}{p_n-p} < 1$, then $\lim_{n\to\infty} \frac{\hat{p}_n - p}{p_n - p} = 0$ where $\hat{p}_n$ comes from Aitken's Δ^2 method. (Hint: let $\delta_n = (p_{n+1}-p)/(p_n-p) - \lambda$ and show that $\lim_{n\to\infty}\delta_n = 0$. Then express $(\hat{p}_{n+1}-p)/(p_n-p)$ in terms of δ_n, δ_{n+1} and λ).*

1

$p_n = 10^{-\alpha^n}$

$p = 0$

$$\frac{|p_{n+1} - p|}{|p_n - p|^\alpha} = \frac{10^{-\alpha^{n+1}}}{(10^{-\alpha^n})^\alpha} = 1$$

$\lambda = 1$

$\lambda < 1$

$p = 0$

$p_1 = 1$

$\frac{p_2}{p_1^\alpha} = \lambda$

$p_2 = \lambda p_1^\alpha = \lambda$

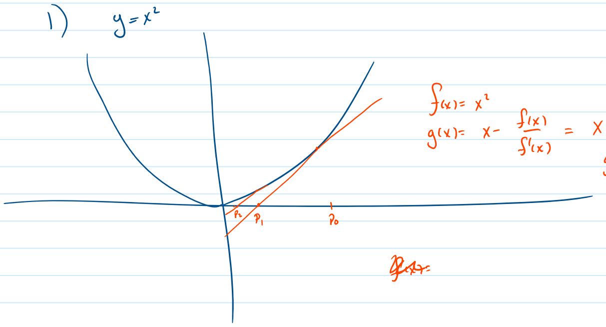
$p_3 = \lambda p_2^\alpha = \lambda \lambda^\alpha = \lambda^{\alpha+1}$

$p_4 = \lambda p_3^\alpha = \lambda(\lambda^{\alpha+1})^\alpha = \lambda \lambda^{\alpha^2+\alpha} = \lambda^{\alpha^2+\alpha+1}$

$p_5 = \dots = \lambda^{\alpha^3+\alpha^2+\alpha+1}$

$$p_n = \lambda^{\sum_{k=0}^{n-2}\alpha^k}$$

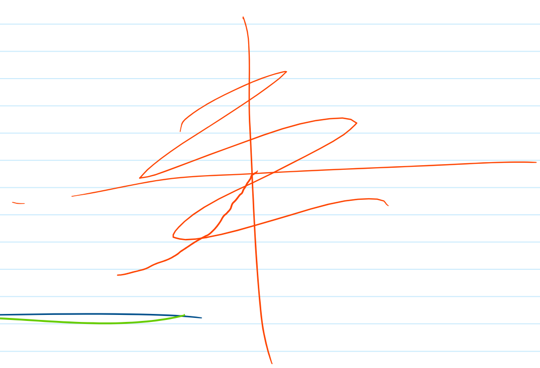
1) $y = x^2$



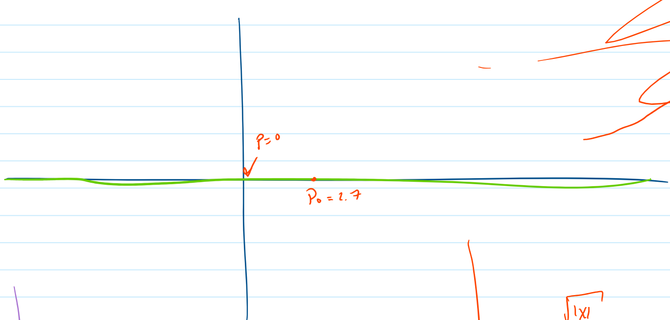
$f(x) = x^2$

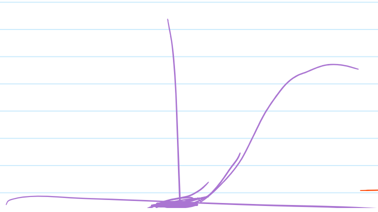
$g(x) = x - \frac{f(x)}{f'(x)} = x - \frac{x^2}{2x} = x - \frac{x}{2} = \frac{x}{2}$

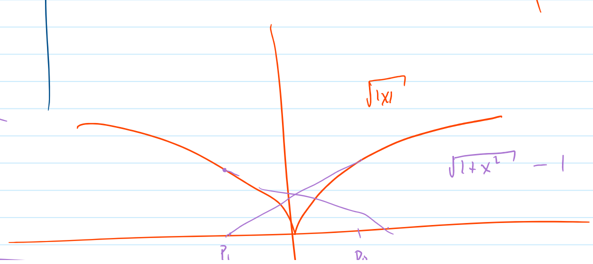
$g' = 1/2$

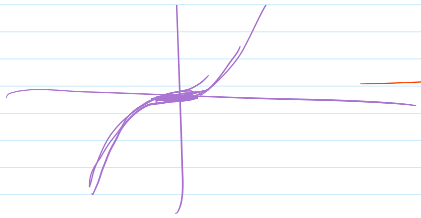


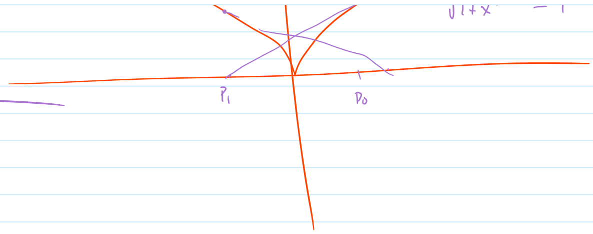
$y = 0$











2) we want $\overbrace{|P - P_n|}$ controlled

$$P_n = P_{n-1} - \frac{f(P_{n-1})}{f'(P_{n-1})}$$

$$|P_n - P_{n-1}| = + \left| \frac{f(P_{n-1})}{f'(P_{n-1})} \right|$$

$$|a+b| \le |a| + |b|$$

$$M = \max f''(x)$$

$$|P_n - P + P - P_{n-1}| \le |P_n - P| + |P - P_{n-1}|$$

$$= |P_n - P_{n-1}|$$

$$g(x) = x - \frac{f(x)}{f'(x)} \qquad \text{then} \qquad x_n = g(x_{n-1})$$

Taylor expand g about $x = p$

$$g(p) = p - \frac{f(p)}{f'(p)} = p$$

$$g'(x) = 1 - \frac{(f'(x))^2 - f(x) f''(x)}{(f'(x))^2}$$

$$g'(p) = 1 - \frac{(f'(p))^2}{(f'(p))^2} = 1 - 1 = 0$$

$$g(x) = p + 0 + \frac{g''(\xi)}{2}(x-p)^2$$

$$P_n - P = g(P_{n-1}) - P$$

$$= P + \frac{g''(\xi)(P_{n-1} - P)^2}{2} - P = \frac{g''(\xi)(P_{n-1} - P)^2}{2}$$

$$|P_n - P| \le \frac{|g''(\xi)|}{2} |P_{n-1} - P|^2$$

$$P_n - P \qquad P_n = P_{n-1} - \frac{f(P_{n-1})}{f'(P_{n-1})}$$

Taylor expand f @ $x = p$

$$f(x) = f(p) + f'(p)(x-p) + \frac{f''(\xi)}{2}(x-p)^2$$

$$= f'(p)(x-p) + \frac{f''(\xi)}{2}(x-p)^2$$

$$f(P_{n-1}) = f'(p)(P_{n-1} - p) + \frac{f''(\xi)}{2}(P_{n-1} - p)^2$$

$\boxed{\text{Taylor @ } x = P_n}$

$$f(x) = f(P_n) + f'(P_n)(x - P_n) + \frac{f''(\xi)}{2}(x - P_n)^2$$

plug in $x = p$

$$0 = f(p) = f(P_n) + f'(P_n)(p - P_n) + \frac{f''(\xi)}{2}(p - P_n)^2$$

$$-\frac{f(P_n)}{\cdots} = (p - P_n) + f''(\xi)\frac{(p - P_n)^2}{\cdots}$$

$$-\frac{f(p_n)}{f'(p_n)} = (p-p_n) + \frac{f''(\xi)\,(p-p_n)^2}{2f'(p_n)}$$

$$p_{n+1} = g(p_n) = \boxed{p_n - \frac{f(p_n)}{f'(p_n)}} = p + \frac{f''(\xi)\,(p-p_n)^2}{2f'(p_n)}$$

$$\left| p_{n+1} - p \right| = \left| \frac{f''(\xi)\,(p-p_n)^2}{2f'(p_n)} \right| \le \frac{M\,|p-p_n|^2}{2\,|f'(p_n)|}$$